Non–standard measure requires non–standard approaches to the integral

(History and recent results on integration with respect to monotone measures)

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Abstract – **After a short history of classical integration, several types of integrals with respect to monotone measures are discussed and exemplified. In particular, we recall the Choquet and the Sugeno integrals, including their axiomatic characterization. The concepts of universal and decomposition integrals are also introduced and exemplified.**

Keywords – **Choquet integral, decomposition integral, Sugeno integral, universal integral, capacity, monotone measure.**

I. Introduction and short history of integration

The earliest traces of formulas which were called much later integration are linked to geometrical objects and their sizes, in particular to the areas of surfaces, lengths of curves or volumes of bodies. Maybe the oldest written trace related to integration can be dated around 1850 BC. The famous Moscow Mathematical Papyrus contains, among others, also a formula for the frustum of a square pyramid (Problem 14). The oldest documented systematic technique related to integration is due to Eudoxus and it is dated around 370 BC. His exhaustion method allows to find areas (lengths, volumes) of considered objects by approximating them by objects whose sizes are known, such as rectangles or triangles, segments, cubes, etc. Several other important results, such as the approximation of perimeter of a circle or volume of a sphere were obtained by similar methods in China (Liu Hui in the 3^{rd} century AD, Zu Geng around 500 AD) and in India (Aryabhata around 500 AD).

The roots of integration not related to the geometry go back to Newton [16] and Leibniz [11], 17^{th} century. Note that the symbol $\int$ is due to Leibniz. Integration of continuous functions was developed significantly by Cauchy [2], beginning of 19^{th} century. The formalization of the standard theory of integration on the real line was introduced in 1854 by Riemann [18]. Still the notion of a measure was somehow hidden and the Riemann integral was built on the volume of a cuboid $[a_1,b_1]\times\ldots\times[a_n,b_n]$ given by $\prod_{i=1}^n(b_i-a_i)$, and the (σ-) additivity of such volumes. The concept of integration with respect to a given σ-additive measure acting on some measurable space was proposed by Lebesgue [9]. Several other generalizations, such as Bochner or Pettis integrals, go out of the framework of real–valued functions, and we will not consider them in this contribution.

The introduction of the Riemann and the Lebesgue integrals is strongly related to the additivity of the underlying measure. For better understanding of the philosophy behind these integrals, let us introduce the next example.

Example 1: Let $f:[0,3]\to\Re$ be given by

$$f(x)=\begin{cases}1 & \text{if } x\in[1,2],\\ 2 & \text{otherwise.}\end{cases}$$

For the Riemann integral, the representation

$$f=2\cdot 1_{[0,1[}+1_{[1,2]}+2\cdot 1_{]2,3]}$$

is crucial and then

$$\int_0^3 f(x)\,dx=2\cdot 1+1\cdot 1+2\cdot 1=5.$$

For the Lebesgue integral with respect to the standard Lebesgue measure λ on $[0,3]$, another representation of f is necessary, namely

$$f=1_{[1,2]}+2\cdot 1_{[0,1[\cup]2,3]},$$

and then

$$\int_{[0,3]} f\,d\lambda=1\cdot 1+2\cdot 2=5.$$

Non–additive concepts in measure theory were considered already in 19^{th} century. For example, Menger [13], an Austrian economist, has considered non–additivity in the context of interacting criteria. The first approach to integration with respect to non–additive measures is due to Vitali [23] in 1925. Considering inner and outer measures, Vitali has proposed to consider (non–negative) functions in the form of their cut systems, i.e., $f:X\to[0,\infty]$ was represented in the form $(\{f\geq t\})_{t\in[0,\infty]}$. This approach was further developed by Choquet [3] in 1953, by Sugeno [22] in 1974, and by many others. For interested readers, we recommend the handbook [17], the monograph [24] or a recent overview paper [8].

Note that the possible interaction is not compatible with the additivity, and thus non–additive measures occur in any branch dealing with subjective decisions and their modeling. Typically, in multicriteria decision support, weights of groups of criteria need not behave additively. Alternatives are usually described by score vector, where each score corresponds to the degree of satisfaction of a considered criteria (on a fixed scale, such as $[0,1]$ or $[0,100]$). The expected utility of such an alternative is then a matter of integration. The aim of this contribution is to discuss such possible integrals with respect to not necessarily additive measure. Note that though we can formally consider any measurable space $(X,\mathcal{A})$ and $\mathcal{A}$–measurable non–negative function $f : X \to [0,\infty]$, we often introduce discussed integrals in their discrete form, i.e., when X is finite (then we identify it with $X = \{1,...,n\}$, n being the cardinality of X), and $\mathcal{A} = 2^X$. We will consider monotone measures $m : \mathcal{A} \to [0,\infty]$, $m(\emptyset) = 0$, $m(X) > 0$, and $m(A) \le m(B)$ whenever $A \subseteq B$ (note that the (σ)–additive measure are not excluded).

In the next section, we recall the Choquet, the Sugeno and the Shilkret integrals. In Section 3, the concept of universal integrals [6] is discussed. Besides this, we deal in Section 3 with decomposition integrals [4], too. Finally, some concluding remarks are added.

II. The Choquet, the Sugeno and the Shilkret integrals

The approach to integration in the case of the Riemann and the Lebesgue integrals is based on partition–based integral sums, i.e., we look first on the domain of the integrand. This approach fails once the additivity is violated, and thus some new technique should be applied. In fact, once starting from the range of the integrand, i.e., exploiting the above mentioned cut–representation of integrands, the additivity does not play any role more. This is exactly the philosophy behind the integrals we recall in this and the next sections.

Consider a monotone measure $m : \mathcal{A} \to [0,\infty]$ and an $\mathcal{A}$–measurable function $f : X \to [0,\infty]$. These two information sources are merged into one non–decreasing real function

$$h_{m,f} : [0,\infty] \to [0,\infty],\ h_{m,f}(t) = m(f \ge t).$$

Note that if m is a probability measure, then $h_{m,f}$ is the survival function related to the random variable f, i.e., the complement to the corresponding distribution function.

Now, we can introduce the next integrals:

- the Choquet integral [3]

$$(C) - \int_X f\, dm = \int_0^\infty h_{m,f}(t)\, dt, \qquad (1)$$

  where the right hand side is the (improper) Riemann integral.
  If $m = P$ is a probability measure, then

$$(C) - \int_X f\, dP = E_P(f) \qquad (2)$$

  is the expected value of the random variable f, and thus the formula (2) allows to compute $E_P(f)$ for any kind of non–negative random variable!
  If $X = \{1,...,n\}$ is finite, then

$$(C) - \int_X f\, dm = \\ = \sum_{i=1}^{n} \left(f((i)) - f((i-1))\right) \cdot m\left(\{(i),...,(n)\}\right), \qquad (3)$$

  where $(.) : X \to X$ is a permutation such that $f((0)) = 0 \le f((1)) \le ... \le f((n))$.
- the Sugeno integral [22]

$$(Su) - \int_X f\, dm = \bigvee_{t\in[0,\infty]} \min\left\{t, h_{m,f}(t)\right\} \qquad (4)$$

  turns in the case $X = \{1,...,n\}$ into

$$(Su) - \int_X f\, dm = \bigvee_{i=1}^{n} \left(f((i)) \wedge m\left(\{(i),...,(n)\}\right)\right). \qquad (5)$$

- the Shilkret integral [21]

$$(Sh) - \int_X f\, dm = \bigvee_{t\in[0,\infty]} t \cdot h_{m,f}(t) \qquad (6)$$

  turns in the case $X = \{1,...,n\}$ into

$$(Sh) - \int_X f\, dm = \bigvee_{i=1}^{n} f((i)) \cdot m\left(\{(i),...,(n)\}\right). \qquad (7)$$

Example 2: Continuing in Example 1, we see that the Choquet integral is related to the representation $f = 1_{[0,3]} + 1_{[0,1[\cup]2,3]}$ and then

$$(C) - \int_{[0,3]} f\, dm = 1 \cdot 3 + 1 \cdot 2 = 5,$$

(note that if m is a σ–additive measure, then the Lebesgue and the Choquet integrals coincide).

Riemann, Lebesgue and Choquet integrals are based on additive integral sums (which have disjoint supports for the first two integrals, but in the Choquet integral case, the supports form a chain). Sugeno and Shilkret integrals are related to maxitive representation (with supports of integral summands forming a chain). In our example this means that:

$$f = 1_{[0,3]} \vee (2 \cdot 1_{[0,1[\cup]2,3]}).$$

Hence

$$(Su) - \int_{[0,3]} f\, dm = (1 \wedge 3) \vee (2 \wedge 2) = 2,$$

while

$$(Sh) - \int_{[0,3]} f\, dm = (1 \cdot 3) \vee (2 \cdot 2) = 4.$$

Recall that two function $f, g : X \to [0,\infty]$ are comonotone whenever they are measurable with respect to a common chain, i.e., if $(f(x) - f(y)) \cdot (g(x) - g(y)) \ge 0$ for any $x, y \in X$. We have the next axiomatic definition of the introduced integrals:

- the Choquet integral is characterized by the comonotone additivity, i.e., it is a non–negative functional satisfying

$$I(f + g) = I(f) + I(g)$$

  whenever f and g are comonotone non–negative $\mathcal{A}$–measurable functions [19], [20]. Note that then

$$I(f) = (C) - \int_X f\, dm,$$

where $m(A) = I(1_A)$, $A \in \mathcal{A}$.

- the Sugeno integral is characterized by the comonotone maxitivity and min–homogeneity [12], i.e.,
$$I(f) = (Su) - \int_X f\, dm$$
for a monotone measure m given by $m(A) = I(1_A)$ whenever $I(f \vee g) = I(f) \vee I(g)$ for any comonotone f and g, and $I(c \wedge f) = c \wedge I(f)$ for any $f, c \in]0, \infty[$.
- the Shilkret integral is characterized by comonotone maxitivity and positive homogeneity, i.e., $I(cf) = c\, I(f)$ for any $c \in]0, \infty[$, see [1].

III. Universal and decomposition integrals

An attempt to classify functionals acting on non–negative measurable functions as integrals has led Klement et al. [6] to the proposal of the concept of universal integrals. Using the notation $\mathcal{M}_{(X,\mathcal{A})}$ for all monotone measures $m : \mathcal{A} \to [0, \infty]$, $\mathcal{F}_{(X,\mathcal{A})}$ for all $\mathcal{A}$–measurable functions $f : X \to [0, \infty]$, and denoting the class of all measurable spaces $(X, \mathcal{A})$ by $\mathcal{D}$, we have the next definition.

Definition 1: A mapping
$$I : \bigcup_{(X,\mathcal{A}) \in \mathcal{D}} \left(\mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})} \right) \to [0, \infty]$$
is called a universal integral whenever for any $(m_i, f_i) \in \mathcal{M}_{(X_i,\mathcal{A}_i)} \times \mathcal{F}_{(X_i,\mathcal{A}_i)}, i = 1, 2$, such that $h_{m_1,f_1} \le h_{m_2,f_2}$ on $]0, \infty]$ it holds
$$I(m_1, f_1) \le I(m_2, f_2) \text{ and } I(m, 1_\emptyset) = 0$$
for any monotone measure m.

Note that the original approach to universal integrals in [6] has considered 3 axioms. Obviously, if $h_{m_1,f_1} = h_{m_2,f_2}$, then $I(m_1, f_1) = I(m_2, f_2)$, compare the fact that an expected value of a random variable depends on the corresponding distribution function only. More, for any $f = c\, 1_A \in \mathcal{F}_{(X,\mathcal{A})}$ and $m \in \mathcal{M}_{(X,\mathcal{A})}$ we have for $t > 0$,
$$h_{m,f}(t) = \begin{cases} m(A) & \text{if } t \le c, \\ 0 & \text{otherwise.} \end{cases},$$
which means that $I(m, c\, 1_A) = c \otimes m(A)$ for some operation $\otimes : [0, \infty]^2 \to [0, \infty]$, independently of the measurable space $(X, \mathcal{A})$. Also, for a fixed $(X, \mathcal{A}) \in \mathcal{D}$, the monotonicity of the universal integral I both in measures and in functions follows. Hence, the above mentioned operation $\otimes$ is increasing in both coordinates and $0 \otimes x = x \otimes 0 = 0$ for any $x \in [0, \infty]$.

All 3 integrals introduced in Section II are universal, and $\otimes$ is the standard product in the case of the Choquet and the Shilkret integrals, while $\otimes = \wedge$ for the Sugeno integral.

Among distinguished universal integrals we recall:

- for fixed $\otimes$, the smallest universal integral $I_\otimes$ is given by
$$I_\otimes(m, f) = \bigvee_{t \in [0,\infty]} \left(t \otimes h_{m,f}(t) \right). \tag{8}$$
Evidently $I_\wedge$ is the Sugeno integral, while $I_\cdot$ is the Shilkret integral.
- when restricting the range of measures and functions to $[0, 1]^2$, for each copula $C : [0, 1]^2 \to [0, 1]$ (an increasing operation on $[0, 1]$ with neutral element $e = 1$, which is supermodular, i.e., $C(\mathbf{x} \vee \mathbf{y}) + C(\mathbf{x} \wedge \mathbf{y}) \ge C(\mathbf{x}) + C(\mathbf{y})$, see [15]) one can introduce a universal integral $I_{(C)}$ by
$$I_{(C)}(m, f) = P_C \left(\left\{ (x, y) \in [0, 1]^2 \,|\, y \le h_{m,f}(x) \right\} \right), \tag{9}$$
where P_C is the probability measure on Borel subsets of $[0, 1]^2$ generated by
$$P_C([0, x] \times [0, y]) = C(x, y),\ x, y \in [0, 1].$$
Observe that for the product copula Π, $\Pi(x, y) = xy$ (copula of independence), $I_{(\Pi)}$ is the Choquet integral. Similarly, for the comonotonicity copula M, $M(x, y) = x \wedge y$, $I_{(M)}$ is the Sugeno integral.

Quite recently, Even and Lehrer [4] have introduced the concept of decomposition integrals. For a fixed $(X, \mathcal{A}) \in \mathcal{D}$, a non–empty finite subset $\mathcal{B} \subset \mathcal{A} \setminus \{\emptyset\}$ is called a collection, and a set $\mathcal{H}$ of some collections is called a decomposition system. Note that a decomposition system $\mathcal{H}$ can be introduced ad hoc for $(X, \mathcal{A})$ fixed, but also by a general rule for an arbitrary $(X, \mathcal{A})$. For example, for $i \in N$, $\mathcal{H}_i = \{\mathcal{B} | \mathcal{B} \subset \mathcal{A} \setminus \{\emptyset\}, 0 < |\mathcal{B}| \le i, \mathcal{B} \text{ is a chain}\}$ is defined for any measurable space $(X, \mathcal{A})$.

Also $\mathcal{H}_\infty$, containing all non–empty finite chains in $\mathcal{A}$, is well defined for any $(X, \mathcal{A})$.

Definition 2: Let $\mathcal{H}$ be a decomposition system on $(X, \mathcal{A})$. Then the decomposition integral
$$I_\mathcal{H} : \mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})} \to [0, \infty]$$
is given by
$$I_\mathcal{H}(m, f) = \\ = \sup \left\{ \sum_i a_i\, m(A_i) | (A_i)_i \in \mathcal{H}, a_i \ge 0, \sum_i a_i 1_{A_i} \le f \right\}. \tag{10}$$
Observe that $I_{\mathcal{H}_1}$ is the Shilkret integral, and $I_{\mathcal{H}_\infty}$ is the Choquet integral.

Considering $\mathcal{H} = \{\mathcal{B} | \mathcal{B} \text{ is a finite } \mathcal{A}\text{–measurable partition of } X\}$, we have that $I_\mathcal{H}$ is Pan–integral of Yang [25], which, if m is σ–additive, coincides with the classical Lebesgue integral.

Similarly, for the maximal decomposition system $\mathcal{H} = \{\mathcal{B} | \mathcal{B} \text{ is a collection } \}$, $I_\mathcal{H}$ is the concave integral recently introduced by Lehrer [10]. This integral is another generalization of the Lebesgue integral and it coincides with the Choquet integral whenever the considered monotone measure m is supermodular.

IV. Concluding remarks

We have discussed several types of integrals. Clearly, more general measures we have considered, more general approaches to integration are needed. Note that once the Riemann integral can be applied, then it coincides with the Lebesgue and with the Choquet integral. Similarly, Choquet integral generalizes the Lebesgue integral and it offers, among others, an alternative unified look on expected values of non–negative random variables. These integrals are based on the standard arithmetics of reals. On the other hand, the Sugeno integral is related to the lattice operations $\wedge$ and $\vee$, while the Shilkret integral is based on the ring $([0, \infty], \vee, \cdot)$. We have recalled also concepts of universal integrals (with new axiomatization) and of decomposition integrals. Observe that the only decomposition integral which can be seen also as universal integrals are integrals $I_{\mathcal{H}_i}, i \in N \cup \{\infty\}$, forming a hierarchical family of integrals

$$\text{Shilkret} = I_{\mathcal{H}_1} \leq I_{\mathcal{H}_2} \leq ... \leq I_{\mathcal{H}_\infty} = \text{Choquet}.$$

For more details see [14].

Several types of universal integrals on $[0,1]$ can be built by means of copulas, see [7]. For further generalizations, for example those based on level–dependent, see [5].

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